SUBJECT: MATHEMATICS

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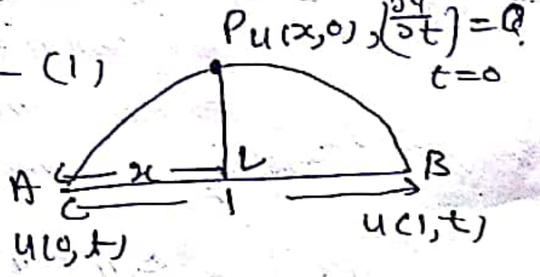
Q6. A string is stretched between the fixed points $(0,0)$ and $(1,0)$ and released at rest from the positions $u = A\sin \pi x$. Find the formula for its subsequent displacement $u(x,t)$.

Solution. The equation of the string is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—(1)}$$



with conditions

$$u(0,t) = 0 \quad \text{—(2)}$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \quad \text{—(3)}$$

$$u(1,t) = 0 \quad \text{—(4)}$$

$$u(x,0) = A\sin \pi x \quad \text{—(5)}$$

The solutions of (1) is given by

$$u(x,t) = (A_1 \cos nx + B_1 \sin nx)(C_1 \cos nct + D_1 \sin nct) \quad \text{—(6)}$$

put $x=0$, we have

$$u(0,t) = (A_1 + 0)(C_1 \cos nct + D_1 \sin nct)$$

$$0 = A_1 (C_1 \cos nct + D_1 \sin nct)$$

$$\Rightarrow A_1 = 0 \quad \text{as } (C_1 \cos nct + D_1 \sin nct) \neq 0$$

Put this value of A_1 in (6), we have

$$u(x,t) = \sin nx\,[B_1 C_1 \cos nct + B_1 D_1 \sin nct) \quad \text{—(7)}$$

$$\frac{\partial u}{\partial t} = \sin nx\,[-nc \cdot B_1 C_1 \sin nct + nc B_1 D_1 \cos nct]$$

put $t=0$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = \sin nx\,[0 + nc B_1 D_1 \cos nct]$$

$$\Rightarrow 0 = \sin nx \cdot B_1 D_1 \cos nct \Rightarrow B_1 D_1 = 0$$

Putting the value of $B_1 D_1$ in equation (7), we get

$$u(x,t) = \sin nx [B_1 C_1 \cos nct] \quad \text{— (8)}$$

put $x = 1$

$u(1,t) = 0 \Rightarrow$

$0 = \sin n . B_1 C_1 \cos nct$

$\Rightarrow \sin n = 0 = \sin m\pi$; $m \in I$, integer

$\Rightarrow n = m\pi$

putting the value of $m\pi$ in (8), we get

$$u(x,t) = \sin m\pi x . B_1 C_1 \cos m\pi ct$$

We can write this as

$$u(x,t) = \sum b_m \sin m\pi x \cos m\pi ct \quad \text{— (9)}$$

where $B_1 C_1 = b_m$, say.

put $t = 0$

$$u(x,0) = \sum_{m=1}^{\infty} b_m \sin m\pi x \quad [\because \cos m\pi c.0 = 1]$$

$$\Rightarrow A \sin \pi x = \sum_{m=1}^{\infty} b_m \sin m x . \pi \quad \text{— (10)}$$

which is a fourier sine series

where $f(x) = A \sin \pi x$

$$b_m = \frac{2}{1} \int_0^1 f(x) \sin m\pi x \, dx$$

$$b_m = \frac{2}{1} \int_0^1 A \sin \pi x \sin m\pi x$$

put $m = 1$

$$b_1 = 2 \int_0^1 A \sin^2 \pi x \, dx \Rightarrow b_1 = 2A \int_0^1 \left(\frac{1 - \cos 2\pi x}{2}\right) dx$$

$$b_1 = A \int_0^1 dx \Rightarrow b_1 = A.$$

if $m = 2, 3, 4, ---$

$b_2 = b_3 = b_4 = --- = 0.$

$$\therefore u(x,t) = \sum b_m \sin m x \pi \cos m\pi ct \Rightarrow$$

$$\boxed{u(x,t) = A \sin \pi x \cos \pi ct}$$ Ans

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7) A string is stretched b/w $(0,0)$ and $(l,0)$ and released at rest from the initial deflection given by

$$f(x) = \begin{cases} \frac{2k}{l}x, & 0<x<\frac{l}{2} \\ \frac{2k}{l}(l-x), & \frac{l}{2}<x<l \end{cases} \quad \text{———}$$

Sol. Given $\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial x^2}\right]$ ——— (1)

conditions: $u(0,t)=0$ — (2) $u(l,t)=0$ — (4)

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \text{ — (3)}$$

& $f(x,0) = \begin{cases} \frac{2k}{l}x, & 0<x<\frac{l}{2} \\ \frac{2k}{l}(l-x), & \frac{l}{2}<x<l \end{cases}$ — (5)

Sol^n of (1) is given by

$u(x,t) = (A_1 \cos nx + B_1 \sin nx)(C_1 \cos nct + D_1 \sin nct)$

$u(0,t)=0 \Rightarrow 0=(A_1+B_1\times 0)(C_1\cos nct + D_1 \sin nct)$ (6)

$\Rightarrow A_1=0$, Put $A_1=0$ into (6), we have

$u(x,t) = E_1 \sin nx \cos nct + F_1 \sin nx \sin nct$ ——— (7)

$\frac{\partial u}{\partial t} = -nc E_1 \sin nx \sin nct + nc F_1 \sin nx \cos nct$

Put $t=0$

$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 + nc F_1 \sin nx \Rightarrow F_1 = 0$, Put $F_1=0$ in (7),

we have $u(x,t) = E_1 \sin nx \cos nct$ ——— (8)

put $x=l \Rightarrow u(l,t) = E_1 \sin nl \cos nct$

(4) $\Rightarrow 0 = E_1 \sin nl \cos nct \Rightarrow \sin nl = \sin m\pi$

(8) & (9) $\Rightarrow$ $\Rightarrow n = \frac{m\pi}{l}$ ——— (9)

$u(x,t) = E_1 \sin\frac{m\pi}{l}x \cos\left(\frac{m\pi}{l}ct\right)$

or, $\sum u_m(x,t) = \sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{l}\cos\left(\frac{m\pi}{l}ct\right)$, for different values of m

$= u(x,t)$ ——— (10)

put $t=0$

$u(x,0) = \sum b_m\left(\frac{\sin m x\pi}{l}\right)$ ——— (11)

which is fourier sine series where $u(x) = f(x)$ = as above

where $b_m = \frac{2}{l}\int_0^l f(x)\sin\frac{m\pi x}{l}\,dx$

$= \frac{2}{l}\int_0^{l/2} f(x)\sin\left(\frac{m\pi x}{l}\right)dx + \frac{2}{l}\int_{l/2}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)dx$

$b_m = \frac{2}{l}\int_0^{l/2}\frac{2k}{l}x\sin\left(\frac{m\pi x}{l}\right)dx + \frac{2}{l}\int_{l/2}^{l}\frac{2k}{l}(l-x)\sin\left(\frac{m\pi x}{l}\right)dx$

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$$b_m = \frac{4k}{l^2}\left[-x\cdot\frac{l}{m\pi}\cos\left(\frac{m\pi x}{l}\right)\Big|_0^{l/2} + \int_0^{l/2} 1\cdot\frac{l}{m\pi}\cos\frac{m\pi x}{l}\,dx\right]$$

$$+\frac{4k}{l^2}\left[-(l-x)\frac{l}{m\pi}\cos\left(\frac{m\pi x}{l}\right)\Big|_{l/2}^{l} + \int_{l/2}^{l}(-1)\frac{l}{m\pi}\cos\left(\frac{m\pi x}{l}\right)dx\right]$$

$$= \frac{4k}{l^2}\left[-\frac{l^2}{2m\pi}\cos\left(\frac{m\pi}{2}\right)+0+\frac{l^2}{m^2\pi^2}\left\{\sin\left(\frac{m\pi x}{l}\right)\right\}_0^{l/2}\right]$$

$$+\frac{4k}{l^2}\left[0+\left(l-\frac{l}{2}\right)\cdot\frac{l}{m\pi}\cos\left(\frac{m\pi}{2}\right)-\frac{l^2}{m^2\pi^2}\left\{\sin\left(\frac{m\pi x}{l}\right)\right\}_{l/2}^{l}\right]$$

$$= -\frac{4k}{2m\pi}\cos\left(\frac{m\pi}{2}\right)+\frac{4k}{m^2\pi^2}\left\{\sin\frac{m\pi}{2}-0\right\}$$

$$+\frac{4k}{l^2}\times\left[\frac{l}{2}\times\frac{l}{m\pi}\cos\frac{m\pi}{2}-\frac{l^2}{m^2\pi^2}\left\{\sin m\pi-\sin\frac{m\pi}{2}\right\}\right]$$

$$= -\cancel{\frac{4k}{2m\pi}\cos\left(\frac{m\pi}{2}\right)}+\frac{4k}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right)$$

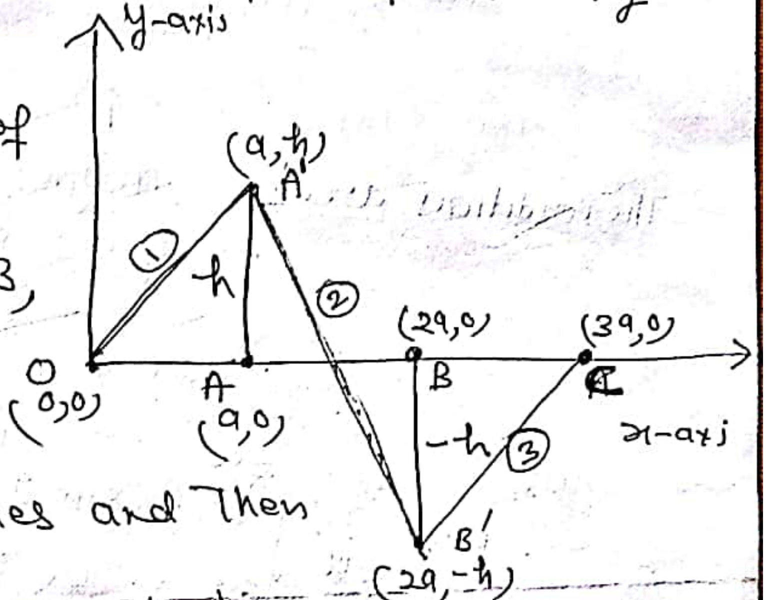
$$+\cancel{\frac{4k}{m\pi}\cos\left(\frac{m\pi}{2}\right)}+\frac{4k}{\cancel{l^2}}\times\frac{\cancel{l^2}}{m^2\pi^2}\sin\frac{m\pi}{2}$$

$$b_m = \frac{8k}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right)$$

$$\boxed{\therefore u(x,t) = \sum_{n=1}^{\infty}\frac{8k}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right)\sin\frac{m\pi x}{l}\cos\left(\frac{m\pi ct}{l}\right)}$$

Ans.

The points of trisection of a string are pulled aside through a distance h on opposite sides of the position of equilibrium, and the string is released from rest. Derive an expression for the displacement of the string at any subsequent time and show that the mid-point of the string always remains at rest.



Sol: Let OABC be the position of equilibrium of the string of length $l = 3a$. Let A and B, the points of trisection of the string be pulled through a distance h on opposite sides and then released.

This is the case in which vibrations are governed by one dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ —(1)

Equ of ~~O(0,0) to~~ OA' $\quad y-0 = \frac{h-0}{a-0}(x-0) \Rightarrow \boxed{y = \frac{h}{a}x}$

Equ of A'B' $\quad y-h = \frac{-h-h}{2a-a}(x-a)$ —(2)

$$a(y-h) = -2hx + 2ha$$

$$ay = -2hx + 3ah = h(3a-2x)$$

$$y = \frac{h}{a}[3a-2x] \quad \text{—(3)}$$

Equ of B'C $\quad y-(-h) = \frac{0-(-h)}{3a-2a}(x-2a)$

$$y+h = \frac{h}{a}(x-2a) \Rightarrow ay + ah = hx - 2ah$$
$$\Rightarrow ay = hx - 3ah = h(x-3a)$$

$$y = \frac{h}{a}[x-3a] \quad \text{—(4)}$$

$\therefore$ y has three ~~two~~ parts as

$$y = f(x,0) = \begin{cases} \frac{h}{a}x & , 0 \le x \le a \\ \frac{h}{a}(3a-2x) & , a \le x \le 2a \\ \frac{h}{a}(x-3a) & , 2a \le x \le 3a \end{cases}$$

Now the wave eqn is $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

The conditions are: $u(0,t)=0$, $u(3a,t)=0$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0.$$

$$\therefore u(x,t) = \Sigma u_m(x,t) = \Sigma b_m \sin\frac{m\pi x}{3a} \cos\frac{m\pi ct}{3a} \quad (10)$$

$l = 3a$, here

where

$$b_m = \frac{2}{3a}\int_0^{3a} f(x) \sin\frac{m\pi x}{3a}\,dx$$

$$b_m = \frac{2}{3a}\int_0^{a} f(x)\sin\frac{m\pi x}{3a}\,dx + \frac{2}{3a}\int_a^{2a} f(x)\sin\frac{m\pi x}{3a}\,dx + \frac{2}{3a}\int_{2a}^{3a} f(x)\sin\frac{m\pi x}{3a}\,dx \quad (11)$$

$$= \frac{2}{3a}(I_1+I_2+I_3)$$

$$I_1 = \int_0^a \frac{h}{a}x \sin\frac{m\pi x}{3a}\,dx$$

$$= \frac{h}{a}\left[-x\cdot\frac{3a}{m\pi}\cos\left(\frac{m\pi x}{3a}\right)\Big|_0^a + \int_0^a \frac{3a}{m\pi}\cos\left(\frac{m\pi x}{3a}\right)dx\right]$$

$$= \frac{h}{a}\left[-\frac{3a}{m\pi}\left\{a\cos\frac{m\pi}{3} - 0\right\} + \frac{9a^2}{m^2\pi^2}\left\{\sin\frac{m\pi x}{3a}\right\}_0^a\right]$$

$$= \frac{h}{a}\left[\frac{-3a^2}{m\pi}\cos\frac{m\pi}{3} + \frac{9a^2}{m^2\pi^2}\times\sin\frac{m\pi}{3}\right]$$

$$I_1 = -\frac{3ha}{m\pi}\cos\frac{m\pi}{3} + \frac{9ha}{m^2\pi^2}\sin\frac{m\pi}{3}$$

$$\frac{2}{3a}I_1 = \frac{2}{3a}\left[-\frac{3ha}{m\pi}\cos\frac{m\pi}{3} + \frac{9ha}{m^2\pi^2}\sin\frac{m\pi}{3}\right]$$

$$\frac{2}{3a}I_1 = -\frac{2h}{m\pi}\cos\frac{m\pi}{3} + \frac{6h}{m^2\pi^2}\sin\frac{m\pi}{3} \quad (12)$$

$$I_2 = \int_a^{2a} f(x) \sin\frac{m\pi x}{3a}\,dx = \int_a^{2a} \frac{h}{a}(3a-2x)\sin\left(\frac{m\pi x}{3a}\right)dx$$

$$= \frac{h}{a}\left[(3a-2x)(-)\frac{3a}{m\pi}\cos\frac{m\pi x}{3a}\Big|_a^{2a} + \int_a^{2a}(-2)\frac{3a}{m\pi}\cos\frac{m\pi x}{3a}dx\right]$$

$$= \frac{h}{a}\left(\frac{3a}{m\pi}\right)\left[(3a-2x)(-)\cos\frac{m\pi x}{3a}\Big|_a^{2a} - 2\int_a^{2a}\cos\frac{m\pi x}{3a}dx\right]$$

$$= \frac{3h}{m\pi}\left[(+a)\cos\frac{2}{3}m\pi + a.\cos\frac{m\pi}{3} - \frac{2.3a}{m\pi}\left[\sin\frac{m\pi x}{3a}\right]_a^{2a}\right].$$

$$= \frac{3ha}{m\pi}\cos\frac{2m\pi}{3} + \frac{3ha}{m\pi}\cos\frac{m\pi}{3} - \frac{3h}{m\pi}\left(\frac{6a}{m\pi}\right)\left[\sin\frac{2m\pi}{3} - \sin\frac{m\pi}{3}\right]$$

$$I_2 = \frac{3ha}{m\pi}\cos\frac{2m\pi}{3} + \frac{3ha}{m\pi}\cos\frac{m\pi}{3} - \frac{18ha}{m^2\pi^2}\left(\sin\frac{2m\pi}{3} - \sin\frac{m\pi}{3}\right)$$

$$\frac{2}{3a}I_2 = \frac{2}{3a}\times\frac{3ha}{m\pi}\cos\frac{2m\pi}{3} + \frac{2}{3a}\cdot\frac{3ha}{m\pi}\cos\frac{m\pi}{3} - \frac{2}{3a}\times\frac{18ha}{m^2\pi^2}\left[\sin\frac{2m\pi}{3} - \sin\frac{m\pi}{3}\right]$$

$$\frac{2}{3a}I_2 = \frac{2h}{m\pi}\cos\frac{2m\pi}{3} + \frac{2h}{m\pi}\cos\frac{m\pi}{3} - \frac{12h}{m^2\pi^2}\left(\sin\frac{2m\pi}{3} - \sin\frac{m\pi}{3}\right) \longrightarrow (13)$$

$$I_3 = \int_{2a}^{3a}\frac{h}{a}(x-3a)\sin\frac{m\pi x}{3a}dx$$

$$= \frac{h}{a}\left[(x-3a)(-)\frac{3a}{m\pi}\cos\frac{m\pi x}{3a}\Big|_{2a}^{3a} + \int_{2a}^{3a}1.\cos\frac{m\pi x}{3a}\cdot\left(\frac{3a}{m\pi}\right)dx\right]$$

$$= \frac{h}{a}\left[\frac{3a}{m\pi}\right]\left[-(x-3a)\cos\frac{m\pi x}{3a}\Big|_{2a}^{3a} + \int_{2a}^{3a}\cos\frac{m\pi x}{3a}dx\right]$$

$$= \frac{3h}{m\pi}\left[0 + (-a)\cos\frac{2m\pi}{3} + \frac{3a}{m\pi}\left(\sin\frac{m\pi x}{3a}\right)_{2a}^{3a}\right]$$

$$I_3 = -\frac{3ha}{m\pi}\cos\frac{2m\pi}{3} + \left(\frac{3h}{m\pi}\right)\left(\frac{3a}{m\pi}\right)\left[\sin m\pi - \sin\frac{2}{3}m\pi\right]$$

$$I_3 = -\frac{3ha}{m\pi}\cos\frac{2m\pi}{3} + \frac{9ha}{m^2\pi^2}\left[0 - \sin\frac{2}{3}m\pi\right]$$

$$\frac{2}{3a}I_3 = \frac{2}{3a}\left(\frac{-3ha}{m\pi}\right)\cos\frac{2m\pi}{3} + \frac{2}{3a}\cdot\frac{9ha}{m^2\pi^2}\left[-\sin\frac{2m\pi}{3}\right]$$

$$\frac{2}{3a}I_3 = \frac{-2h}{m\pi}\cos\frac{2m\pi}{3} + \frac{6h}{m^2\pi^2}\left(-\sin\frac{2m\pi}{3}\right) \longleftarrow (14)$$

Adding (12), (13) and (13), we have

$$b_m = \frac{2}{3a}(I_1 + I_2 + I_3), \text{ we have}$$

$$b_m = \cancel{-\frac{2h}{m\pi}\cos\frac{m\pi}{3}} + \frac{6h}{m^2\pi^2}\sin\frac{m\pi}{3} + \frac{2h}{m\pi}\cos\frac{2m\pi}{3}$$

$$\cancel{+\frac{2h}{m\pi}\cos\frac{m\pi}{3}} - \frac{12h}{m^2\pi^2}\sin\frac{2m\pi}{3} + \frac{12h}{m^2\pi^2}\sin\frac{m\pi}{3}$$

$$-\frac{2h}{m\pi}\cos\frac{2}{3}m\pi - \frac{6h}{m^2\pi^2}\sin\frac{2}{3}m\pi$$

$$b_m = \left(\frac{6h}{m^2\pi^2} + \frac{12h}{m^2\pi^2}\right)\sin\frac{m\pi}{3} - \frac{18h}{m^2\pi^2}\sin\frac{2}{3}m\pi$$

$$= \frac{18h}{m^2\pi^2}\left[\sin\frac{m\pi}{3} - \sin\frac{2m\pi}{3}\right]$$

$$= \frac{18h}{m^2\pi^2}\left[\sin\frac{m\pi}{3} - \sin\left(m\pi - \frac{m\pi}{3}\right)\right]$$

$$= \frac{18h}{m^2\pi^2}\left[1 + (-1)^m\right]\sin\frac{m\pi}{3}$$

$$b_m = \frac{18h}{m^2\pi^2}\begin{cases} 0, & \text{if } m \text{ is odd} \\ 2.\sin\frac{m\pi}{3}, & \text{if } m \text{ is even} \end{cases}$$

$$\therefore b_m = \frac{36h}{m^2\pi^2}\sin\frac{m\pi}{3} \quad \text{if } m \text{ is even,}$$

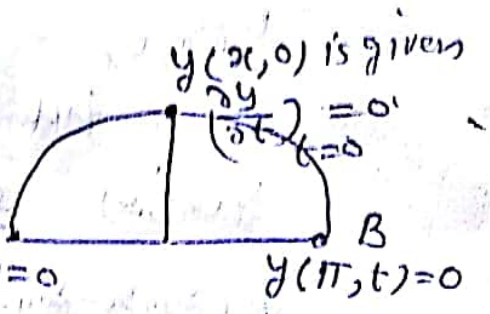
put $m = 2k$

$$b_{2k} = \frac{36h}{4k^2\pi^2}\sin\frac{2k\pi}{3}, \quad k = 1, 2, \ldots$$

$$U(x,t) = \sum_{k=1}^{\infty}\frac{36h}{4k^2\pi^2}\sin\frac{2k\pi}{3}.\cos\frac{2k\pi x}{3}\sin\frac{2k\pi x}{3a}$$

Q9. Find the deflection $u(x,t)$ of the vibrating string of length $l = \pi$, ends are fixed and $c^2 = 1$. Initial velocity is zero and initial deflection is

$$f(x,0) = k(\sin x - \sin 2x).$$



Solⁿ. The Equation of the String is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \left[\frac{\partial^2 y}{\partial x^2}\right]$$

or, $$\frac{\partial^2 y}{\partial t^2} = 1 \cdot \frac{\partial^2 y}{\partial x^2} \quad \text{—— (1)} \quad [\because c^2 = 1, \text{ given}]$$

The conditions are

$$y(0,t) = 0 \quad \text{— (2)}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{— (3)}$$

$$y(\pi,t) = 0 \quad \text{— (4)}$$

$$y(x,0) = k(\sin x - \sin 2x) \quad \text{— (5)}$$

The solution of (1) is given by

$$y(x,t) = (A_1 \cos nx + B_1 \sin nx)(C_1 \cos nt + D_1 \sin nt) \quad \text{— (6)}$$

put $x = 0$

$\Rightarrow y(0,t) = (A_1 + 0)(C_1 \cos nt + D_1 \sin nt)$

$\Rightarrow 0 = A_1$, as $(C_1 \cos nt + D_1 \sin nt) \neq 0$

putting $A_1 = 0$ in (6), we get

$$y(x,t) = \sin nx(B_1 C_1 \cos nt + B_1 D_1 \sin nt) \quad \text{— (7)}$$

$$\frac{\partial y}{\partial t} = \sin nx[-nB_1C_1 \sin nt + nB_1D_1 \cos nt]$$

put $t = 0$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sin nx\,[0 + nB_1D_1] \Rightarrow B_1D_1 = 0.$$

putting the value of $B_1 D_1$ into (7), we get

$y(x,t) = \sin nx\,[B_1 C_1 \cos nt]$

$y(x,t) = B_1 C_1 \sin nx \cos nt$ —— (8)

put $x = \pi$

$y(\pi,t) = 0 \Rightarrow 0 = B_1 C_1 \sin n\pi \cos nt$

$\Rightarrow \sin n\pi = 0$ as $B_1 C_1 \neq 0$ & $\cos nt \neq 0$

$\Rightarrow \sin n\pi = 0 = \sin m\pi$

$\Rightarrow n = m\pi$; m is an integer

∴ (8) can be written as

$$y(x,t) = \sum_{m=1}^{\infty} y_m(x,t) = \sum_{m=1}^{\infty} B_m \sin mx \cos mt$$

$$\Rightarrow y(x,t) = \sum_{m=1}^{\infty} B_m \sin mx \cos mt \quad \text{—— (9)}$$

$y(x,0) = k(\sin x - \sin 2x) \Rightarrow$ [∵ $\cos 0 = 1$]

$$k(\sin x - \sin 2x) = \sum_{m=1}^{\infty} B_m \sin mx \quad \text{—— (10)}$$

which is a Fourier sine series where

$y(x) = k(\sin x - \sin 2x)$

& $B_m = \frac{2}{\pi}\int_0^{\pi} y(x) \sin mx\,dx$

$= \frac{2}{\pi}\int_0^{\pi} k(\sin x - \sin 2x)\sin mx\,dx$

$$B_m = \frac{2k}{\pi}\int_0^{\pi} \sin x \sin mx\,dx - \frac{2k}{\pi}\int_0^{\pi} \sin 2x \sin mx\,dx \quad \text{—— (11)}$$

put $m = 1$, we have

$B_1 = \frac{2k}{\pi}\int_0^{\pi} \sin^2 x\,dx \Rightarrow B_1 = \frac{2k}{\pi}\int_0^{\pi} \frac{1-\cos 2x}{2}\,dx$

$= \frac{2k}{\pi}\cdot\frac{1}{2}\cdot\pi = k$

$\Rightarrow B_1 = k$

put $m=2$

$$B_2 = \frac{2k}{\pi}\int_0^{\pi} \sin x \sin 2x \, dx - \frac{2k}{\pi}\int_0^{\pi} \sin^2 2x \, dx$$

$$= 0 - \frac{2k}{\pi}\int_0^{\pi}\left(\frac{1-\cos 4x}{2}\right)dx$$

$$= -\frac{2k}{\pi} \times \frac{1}{2} \times \pi \Rightarrow B_2 = -k$$

put $m = 3, 4, 5, \ldots$

We get $B_3 = 0 = B_4 \ldots$

as $\int_0^{\pi} \sin x \sin mx \, dx = 0$ if $m \geq 2$

& $\int_0^{\pi} \sin 2x \sin mx \, dx = 0$ if $m \geq 3$.

$\therefore$ (9) $\Rightarrow$

$$y(x,t) = B_1 \sin x \cos t + B_2 \sin 2x \cos 2t$$

$$\boxed{y(x,t) = k \sin x \cos t - k \sin 2x \cos 2t}$$

The Sturm-Liouville Problem [SLP]

The problem of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(\ )y = f(x), \quad p(x) > 0, \ q(x) \text{ is real}$$

, for $a \leq x \leq b$

subject to boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real and each is non zero

, is called Sturm Liouville problem,

Examples on Sturm Liouville

(1) Solve $y'' + \lambda y = 0$, $y(0) = 0$
$y(l) = 0$

Solution. Given

$$y'' + \lambda y = 0 \quad \text{--- (1)}$$

with initial conditions

$$y(0) = 0 \quad \text{--- (2)}$$
$$y(l) = 0 \quad \text{--- (3)}$$

Case I $\lambda = 0$

Equ (1) $\Rightarrow$ $y'' = 0 \Rightarrow \frac{dy}{dx} = A_1$

$$y(x) = y = A_1 x + B_1$$

$y(0) = 0 \Rightarrow 0 = A_1 \cdot 0 + B_1 \Rightarrow B_1 = 0$

$y(l) = 0 \Rightarrow 0 = A_1 l + B_1 \Rightarrow A_1 = 0$

$\therefore$ $y(x) = 0 \cdot x + 0$ when is not a solution because $y(x)$ must be non zero.

Case II $\lambda = -n^2$, $n > 0$

then equation (1) $\Rightarrow$

$$y'' - n^2 y = 0$$

Auxiliary equation is given by

$$D^2 - n^2 = 0 \Rightarrow D = \pm n$$

$\therefore$ $y(x) = A_1 e^{nx} + B_1 e^{-nx}$

$y(0) = 0 \Rightarrow 0 = A_1 + B_1 \Rightarrow B_1 = -A_1$

$y(l) = 0 \Rightarrow 0 = A_1 e^{nl} + B_1 e^{-nl}$

$\Rightarrow 0 = A_1 e^{nl} - A_1 e^{-nl}$

$\Rightarrow 0 = A_1 [e^{nl} - e^{-nl}]$

$\Rightarrow A_1 = 0$ as $e^{nl} - e^{-nl} \neq 0$

$\Rightarrow B_1 = 0$ $[\because B_1 = -A_1]$

$\therefore$ $y(x) = 0 \cdot e^{nx} + 0 e^{-nx} = 0$, when is not an eigen function.

Case III $\lambda = n^2$, $n > 0$ then equation (1)

$\Rightarrow$ $(D'' + n^2)y = 0$

A.E. is $D'' + n^2 = 0 \Rightarrow D = \pm i n$

$\therefore$ $y(x) = A_1 \cos nx + B_1 \sin nx$

$y(0) = 0 \Rightarrow 0 = A_1$

$y(l) = 0 \Rightarrow 0 = A_1 \cos nl + B_1 \sin nl$

$\Rightarrow 0 = B_1 \sin nl$ as $A_1 = 0$

if $B_1 = 1$ then $\sin nl = \sin m\pi$

$\Rightarrow n = \frac{m\pi}{l}$

$\therefore$ For eigen values $\lambda_n = n^2 = \frac{m^2\pi^2}{l^2}$

The required eigen vectors (or eigen function) is given by

$y(x) = \sin \frac{m\pi}{l}$, m is an integer

(2) Solve $X''(x) - \lambda X(x) = 0$, $0 \le x \le 1$

$X(0) = 0$, $X'(1) = -3X(1)$

Soln Given S.L.P is

$$X''(x) - \lambda X(x) = 0 \quad \text{—— (1)}$$

with initial conditions as

$$X(0) = 0 \quad \text{—— (2)}$$

$$X'(1) = -3X(1) \quad \text{—— (3)}$$

Case I $\lambda = 0$ so equation (1) $\Rightarrow$

$X''(x) = 0 \Rightarrow X'(x) = A$

$\Rightarrow X(x) = Ax + B$

$X(0) = 0 \Rightarrow B = 0$,

$X'(x) = A$

$X'(1) = 3X(1)$

$\Rightarrow$ $A = 3[A + B] \Rightarrow A = 3[A + 0]$ $[\because B = 0]$

$\Rightarrow A = 0$

$\therefore$ $X(x) = 0 \cdot x + 0 = 0$

which is not an eigen function

Case II $\lambda = +n^2$, Equ (1) $\Rightarrow$

$$X'' - n^2 X = 0$$

A.E is $D'' - n^2 = 0 \Rightarrow D = \pm n$

$\therefore X(x) = A e^{-nx} + B e^{nx}$

$X(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A$

Now, $X(1) = A e^{-n} + B e^{n}$

$X'(x) = -n e^{-nx} A + B n e^{nx}$

Now $X'(1) = -n e^{-n} A + B n e^{n}$

Now, $X'(1) = -3X(1) \Rightarrow$

$-n e^{-n} A + B n e^{n} = -3[A e^{-n} + B e^{n}]$

$\Rightarrow -n e^{-n} A + B n e^{n} = -3A e^{-n} - 3B e^{n}$

$\Rightarrow (3A - nA) e^{-n} + B n e^{n} + 3B e^{n} = 0$

$\Rightarrow (3A - nA) e^{-n} - A n e^{n} - 3A e^{n} = 0$ as $B = -A$

$\Rightarrow A[3-n] e^{-n} - A[n e^{n} + 3 e^{n}] = 0$

$\Rightarrow A[(3-n) e^{-n} - (n e^{n} + 3 e^{n})] = 0$

$\Rightarrow A = 0 \Rightarrow B = 0$ as $B = -A$

$\therefore X(x) = 0 \cdot e^{-nx} + 0 \, e^{nx} = 0$, which is not an eigen function

Case III $\lambda = -n^2$, $n > 0$

$\therefore$ equation (1) $\Rightarrow$

$X'' + n^2 X = 0 \Rightarrow$ A.E is $D^2 + n^2 = 0$

$\Rightarrow D = \pm i . n$

$\therefore X(x) = A \cos nx + B \sin nx$

$X(0) = 0 \Rightarrow A = 0$

$\therefore X(x) = B \sin nx$

$X'(x) = B n \cos nx$

$X'(1) = B n \cos n$ & $X(1) = B \sin n$.

$X'(1) = -3 X(1) \Rightarrow$

$B n \cos n = -3 B \sin n$

$\Rightarrow B[n \cos n + 3 \sin n] = 0$ if $B \neq 0$ then

$n \cos n = -3 \sin n \Rightarrow \tan n = -\frac{n}{3}$

$\therefore \lambda_n = -n^2$, $n = \tan^{-1}\left(-\frac{n}{B}\right)$

• $\therefore X(x) = B \sin nx$,

which is the eigen function

(3) Solve the differential equation

$$x^2 y'' + 2xy' - 6y = 0$$

Soln. Given differential eqn is

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 6y = 0 \quad \text{--- (1)}$$

Transforming $x \to z$ by letting

$x = e^z$ eqn (1) $\Rightarrow$

$\Rightarrow z = \log x$

$$[D'(D'-1) + 2D' - 6]y = 0 \quad , \quad D' \equiv \frac{d}{dz}$$

A.E is

$$D'^2 - D' + 2D' - 6 = 0$$

$$\Rightarrow D'^2 + D' - 6 = 0$$

$$(D'+3)(D'-2) = 0$$

$$\Rightarrow D' = -3, 2$$

$$\therefore y = Ae^{-3z} + Be^{2z}$$

$$y(x) = y = A \cdot \frac{1}{x^3} + B \cdot x^2$$

$$y(x) = y = Ax^{-3} + Bx^2$$

Q4 Solve $x^2 y'' + 2xy' + \lambda y = 0$, $\lambda > \frac{1}{4}$

Soln. Given differential eqn is

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + \lambda y = 0 \quad \text{--- (1)}$$

changing $x \to z$ by putting $x = e^z$

eqn (1) $\Rightarrow$

$$D'(D'-1) + 2D' + \lambda = 0, \quad D' \equiv \frac{d}{dz}$$

$$\Rightarrow D'^2 - D' + 2D' + \lambda = 0$$

$$\Rightarrow D'^2 + D' + \lambda = 0$$

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$$\Rightarrow D' = \frac{-1 \pm \sqrt{1-4\lambda}}{2}$$

$$\Rightarrow D' = \left(-\frac{1}{2}\right) \pm \frac{1}{2} \cdot 2\sqrt{(-)\left(\lambda-\frac{1}{4}\right)}$$

$$\Rightarrow D' = \left(-\frac{1}{2}\right) \pm i\sqrt{\lambda-\frac{1}{4}}$$

$$\therefore y(x) = e^{-\frac{1}{2}z}\left[A\cos\left(\sqrt{\lambda-\frac{1}{4}}\right)z + B\sin\sqrt{\lambda-\frac{1}{4}}\,z\right]$$

$$, z = \log x$$

$$y(x) = x^{-\frac{1}{2}}\left[A\cos\sqrt{\lambda-\frac{1}{4}}\cdot\log x + B\sin\sqrt{\lambda-\frac{1}{4}}\cdot\log x\right]$$

which is the required solution.